Refs

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6. Supersymmetry (SUSY). Simple example — Witten's model.

In formalism of second quantization bosons and fermions turned out to be very similiar (at least from the point of view of mathematical description)

Bosons (b) fermions (f)

$$[b_\alpha, b_\beta^+] = \delta_{\alpha\beta} \qquad \{f_\alpha, f_\beta^+\} = \delta_{\alpha\beta}$$

(instead of commutators for bosons we have anticommutators for fermions; with this replacement plus different normalization factors many formulas look very alike).

Electrons and phonons are highly different quasiparticles. Nevertheless in second quantization form their Hamiltonian look almost identical. We do not understand the deep physical reasons why bosons and fermions have much in common. For a long time these two types of particles (they belong to different unitary representations of Lorentz group) were considered as completely different. Last 20 years there was a progress in better understanding of common features of bosons and fermions. A new symmetry (supersymmetry) was proposed to connect particles with different statistics. If we assume the existance of supersymmetry in the Nature then it is possible to explain above similiarity between bosons and fermions. Many interesting ideas were developed in this direction however physicists are still far from deep and clear understanding of this fundamental problem.

Let us adopt the standpoint that similiarity between particles with integer and halfinteger spin is due to some symmetry of the Nature. Then fermions and bosons should belong to irreducible representations of the group of supersymmetry. They can share many common properties.

Here we want to understand how this symmetry looks like. In real world supersymmetry looks rather complicated due to spinoral structure of some of their generators. However the basic ideas of SUSY are simple and can be understood by considering a toy model — supersymmetric quantum mechanics (Witten, 1981?)

Consider a state vector $|n_B, n_F\rangle$ (in n-representation) which describes the quatum state with the number of bosons n_B ($n_B = 0, 1, 2, \ldots$) and fermions — $n_F = 0, 1$. Bosonic creation operator b^+ and fermionic destruction operator f act on this state in a standard way

$$b^+ |n_B, n_F\rangle \sim |n_B + 1, n_F\rangle$$

$$f |n_B, n_F\rangle \sim |n_B, n_F - 1\rangle$$

We want to construct operators which transform bosons to fermions and vice a versa

$$Q_+ |n_B, n_F\rangle \sim |n_B - 1, n_F + 1\rangle$$

$$Q_- |n_B, n_F\rangle \sim |n_B + 1, n_F - 1\rangle$$

In terms of "b" and "f" (bosonic and fermionic operators)

The new operators $Q_\pm$ will look like

(48) $Q_+ = q b f^+$, $Q_- = q b^+ f$ (q is real number)

$(Q_-)^+ = Q_+$

Since $(f^+)^2 = (f^-)^2 = 0 \implies Q_+^2 = Q_-^2 = 0$

If we introduce linear combinations of Q_+ and Q_-

$$\left.\begin{aligned} Q_1 &= Q_+ + Q_- \\ Q_2 &= -i(Q_+ - Q_-) \end{aligned}\right\} \quad Q_1^2 = \underbrace{Q_+^2}_{=0} + \underbrace{Q_-^2}_{=0} + \{Q_+, Q_-\}$$

$\{a,b\}$ - anticommutator

$Q_\pm = \frac{1}{2}(Q_1 \pm i Q_2) \Longleftrightarrow Q_1^+ = Q_1 \quad Q_2^+ = Q_2$ (Hermitian operato.

~~then~~

(49) $\boxed{Q_1^2 = Q_2^2} = \{Q_+, Q_-\}$, $\{Q_1, Q_2\} = 0$

Now one can guess the form of the simplest supersymmetric Hamiltonian

(50) $H = Q_1^2 = Q_2^2 = \{Q_+, Q_-\}$

($[H,Q] = 0$ Q is any of the operators: $Q_{1,2}$ or $Q_\pm$

It means that if $|\psi\rangle$ is eigenstate of the Hamiltonian H then $Q|\psi\rangle$ is also eigenstate with the same eigenvalue. $Q_\pm$ transforms boson state to fermion one and vice versa. Hence supersymmetric transformations leave the Hamiltonian Eq(s) invariant)

We ~~got to~~ the simplest superalgebra

(51) $$\boxed{\begin{aligned} &[H, Q_i] = 0 \\ &\{Q_i, Q_j\} = 2\delta_{ij} H \end{aligned}}$$

Z_2 - graded algebra contains two types of operators ("even" and "odd". In our case
even (E) $\Longleftrightarrow$ H
odd (O) $\Longleftrightarrow$ Q

The structure of superalgebra always ~~alw~~ looks like

(i) $[E, E] \sim E$ The line (ii) means that

(ii) $[E, \text{"O"}] \sim \text{"O"}$ odd operators belong to some

(iii) $\{\text{"O"}, \text{"O"}\} \sim E$ representation of ordinary (not super) Lee algebra

Superalgebra in real (3+1)-dimensional space-time:
In real world Odd generators (superchargess) should have spinoral structure (fermions and bosons have different spins). Superalgebra is the extension to "Odd" generators the ~~usual~~ ordinary Poincare algebra. The last consists of 10 generators

Translations $\hat{P}_\mu$ $(\mu = 0,1,2,3)$ $(\mu = 0 \; \hat{P}_0 \equiv \hat{H})$ (4)

Rotations in 3+1 space-time $\hat{J}_{\mu\nu} = -\hat{J}_{\nu\mu}$ +

(because of antisymmetry the number of generators

$(4\cdot 4 - 4)/2 = 6$ (6)

diagonal

10

New "Odd" generators are Q_α $(\alpha = 1,2,3,4)$ (4)

$$\begin{cases} [P_\mu, Q_\alpha] = 0 \\ [J_{\mu\nu}, Q_\alpha] = \frac{i}{2}(\sigma_{\mu\nu})_{\alpha\beta} Q_\beta \\ \{Q_\alpha, Q_\beta\} = -\underbrace{(\gamma_\mu \hat{C})_{\alpha\beta}}_{\text{symmetric matrix}} P_\mu \end{cases}$$

$\sigma_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu]$

antisymmetric matrix

γ_μ are Dirac (4×4) matrices

Here $\hat{C}$ is the charge conjugation operator

$\hat{C}^{-1}\gamma_\mu C = -\gamma_\mu^+$

Properties of SUSY

1. Spectrum of supersymmetric Hamiltonian is always nonnegative

$\{ H = Q_{1,2}^2 \qquad Q$ is hermitian operator $\Rightarrow (\text{eigenvalue})^2 \geq 0$

2. Degeneracy of energy spectrum $(E>0)$

For N supercharges (N - even) $\quad \nu = 2^{N/2}$

Example: $N=2$ Supercharges $Q_1, Q_2 \quad [H, Q_i] = 0$

$$\{Q_1, Q_2\} = 0$$

Let ψ_1 is the eigenfunction both of H and Q_1 $\Rightarrow$

$Q_1\psi_1 = q\psi_1 \quad , \quad H\psi_1 = Q_1^2\psi_1 = q^2\psi_1$

Then $\psi_2 = Q_2\psi_1$ is also eigenfunction of H with the same energy q^2

The "proof" is trivial $\Rightarrow$ $H\psi_2 = H\underbrace{Q_2\psi_1}_{\psi_2} = Q_2H\psi_1 =$

$$= q^2\underbrace{Q_2\psi_1}_{\psi_2}$$

States ψ_1 and $Q_2\psi_1$ are usually called superpartners

$(Q_1\psi_1 = q\psi_1 \qquad Q_1\psi_2 = Q_1(Q_2\psi_1) = -Q_2Q_1\psi_1 = -q\psi_2)$

Supersymmetric Oscillator

Let us find Hamiltonian for our supercharges Eq. (48)

$$\hat{H} = \{Q_+, Q_-\} = q^2(bf^+b^+f + b^+fbf^+) = q^2\{(b^+b + \tfrac{1}{2}) + f^+f - \tfrac{1}{2}\}$$

$bb^+f^+f = (1+b^+b)f^+f \qquad b^+b(ff^+) = b^+b(1-f^+f) \qquad \tfrac{1}{2} - \tfrac{1}{2} = 0.$

$$(52)\quad \hat{H} = q^2\Big\{\underbrace{\big(b^+b+\tfrac{1}{2}\big)}_{\text{Bosonic oscillator}} + \underbrace{\big(f^+f-\tfrac{1}{2}\big)}_{\text{fermionic oscillator}}\Big\}$$

$\hat{n}_B \Rightarrow 0,1,2,\ldots$ $\quad\quad \hat{n}_F \Rightarrow 0,1.$

The vacuum energy of SUSY oscillator $E_0(n_F=n_B=0)=0$!

(In QFT vacuum energy is divergent quantity

scalar boson fields zero-point fluctuations $\oplus\frac{\hbar}{2}\sum_{k=0}^{\infty}\omega_B(\vec{k})$ positive

Fermions (Dirac) $-(2s+1)\sum_{k=0}^{\infty}\hbar\omega_F(\vec{k})$ negative

s – spin

SUSY $\Rightarrow$ $\omega_B(\vec{k}) = \omega_F(\vec{k})$

number of bosons degree of freedom =
= number of chiral Majorana fermions

If SUSY is not spontaneously broken $\Rightarrow$ $\boxed{E_{Vacuum} = 0}$

SUSY solves the problem of (most severe) vacuum energy divergency. Unfortunately it can not solve the problem of all UV divergencies inherent to QFT)

Physical realization of SUSY oscillator is Pauli's Quantum Mechanics in 2D space for the case of uniform magnetic field (Landau quantization of cyclotron orbits)

(53) Pauli's Hamiltonian $\left[\hat{H} = \frac{(\vec{p}-\frac{e}{c}\vec{A})^2}{2m} + \frac{e\hbar}{2mc}\vec{\sigma}\vec{B}\right.$

$\vec{B} = \nabla\times\vec{A}$ ($\vec{A}$ – vector-potential of external e.m. field)

We will assume that masses entering diamagnetic and paramagnetic parts of $\hat{H}$ are equal (usually it is not the case for electrons in solids, Direct magnetic field along z-axis, then

$$\vec{\sigma}\vec{B} \Rightarrow \sigma_3 B \Rightarrow \pm B \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In uniform magnetic field cyclotron motion is quantized (Landau)

$$\frac{(\hat{\vec{p}} - e\vec{A})^2}{2m} \Rightarrow \underbrace{\frac{eB\hbar}{mc}}_{\hbar\omega_B}\left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2 \ldots$$

(54) $$\hat{H} \Rightarrow E_{n,\pm} = \hbar\omega_B\Big(\underbrace{n + \tfrac{1}{2}}_{\text{bosonic part}} \quad \underbrace{\pm\tfrac{1}{2}}_{n_F - \frac{1}{2}}\Big) \qquad n_F - \tfrac{1}{2} = \begin{cases} \tfrac{1}{2} & n_F = 1 \\ -\tfrac{1}{2} & n_F = 0 \end{cases}$$

Double degeneracy of energy levels ($E > 0$) is direct consequence of (hidden) SUSY of 2D Pauli's quantum mechanics (for uniform magnetic field all levels are additionally infinitely degenerated $\nu = eB/2\pi\hbar c$).

Occupation number for fermions can take only two values $n_F = 0$ or 1. It is useful to represent operators f^+, f as linear combinations of Pauli matrices

$$f^+ = \sigma^+ \qquad f = (f^+)^+ = \sigma^- \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma^\pm = \tfrac{1}{2}(\sigma_x \pm i\sigma_y)$$

$$\underbrace{\hat{n}_F}_{\text{fermion occupation number}} = f^+f^- = \sigma^+\sigma^- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$n_F = 1$ $\quad$ $n_F = 0$

$$(\sigma^+)^2 = (\sigma^-)^2 = 0$$

Introduce boson operators $\hat{B}^\pm = \hat{B}_1 \mp i\hat{B}_2$

Recall that for boson oscillator $a \sim (\hat{q} + i\hat{p})$ "coordinate" "momentum"

Then (see Eq. 48)

$$Q_1 = Q_+ + Q_- = \hat{b}_1 \sigma_1 + \hat{b}_2 \sigma_2 \qquad \Rightarrow \qquad \{Q_1, Q_2\} = 0$$
$$Q_2 = -i(Q_+ - Q_-) = \hat{b}_1 \sigma_2 - \hat{b}_2 \sigma_1 \qquad\qquad Q_1^2 = Q_2^2$$

(Q_j – supercharges)

Model (SUSY) Hamiltonian

$$\hat{H} = Q_1^2 = Q_2^2 = \frac{1}{2}\{\hat{b}^-, \hat{b}^+\} + \frac{1}{2}[\hat{b}^-, \hat{b}^+]\sigma_3 =$$
$$= \begin{pmatrix} \hat{b}^-\hat{b}^+ & 0 \\ 0 & \hat{b}^+\hat{b}^- \end{pmatrix}$$

We want to have Hamiltonian quadratic ~~in~~ momentum.

Ansatz

$$b^{\pm} = \frac{1}{\sqrt{2}}\left(W(q) \mp i\hat{p}\right)$$

q – coordinate

if $W(q) = q$ (SUSY oscillator) Eq. (52), (54)

<u>1D N=2 supersymmetric QM</u> (E. Witten)

$$(55) \qquad \boxed{\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}W^2(q) \pm \frac{1}{2}\sigma_3 W'(q)}$$

$W(q)$ sometimes is called superpotential, $W' \equiv \frac{dW(q)}{dq}$

For <u>any</u> function $W(q)$ we have N=2 (two "odd" generators – supercharges) SUSY structure of Hamiltonian. It means that all energy levels ($E>0$) are <u>doubly degenerated</u> (it is a mere consequence of the existance of two anticommuting integrals of motion $Q_{1,2}$)

The zero-energy level (if it exists), of course, is not degenerated (we are studying 1D quantum mechanics!!!). We show that the existance of zero energy (vacuum) level is determined by <u>topological</u> <u>properties</u> of superpotential $W(x)$.

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \qquad \begin{aligned} H_+ &= \delta^- \delta^+ \\ H_- &= \delta^+ \delta^- \end{aligned}$$

We are looking for solution $H_+ \psi_+ = 0$ $\quad \psi = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}$

$H\psi = 0$

or $H_- \psi_- = 0$ $\quad \psi = \begin{pmatrix} \psi_- \\ 0 \end{pmatrix}$

Our problem is reduced to solving differential eq. of first order

$$\delta^+ \psi_+ = 0 \quad \text{or} \quad \delta^- \psi_- = 0$$

$$\Downarrow$$

$$\left(\frac{d}{dx} \mp W(x)\right) \psi_\pm(x) = 0$$

$$\Downarrow$$

$$\psi_\pm \sim \exp\left\{ \pm \int_0^x W(x')\,dx' \right\}$$

To describe physical states $\psi_\pm$ must be normalizable

$$\int_{-\infty}^{\infty} |\psi_\pm(x)|^2 dx < \infty$$

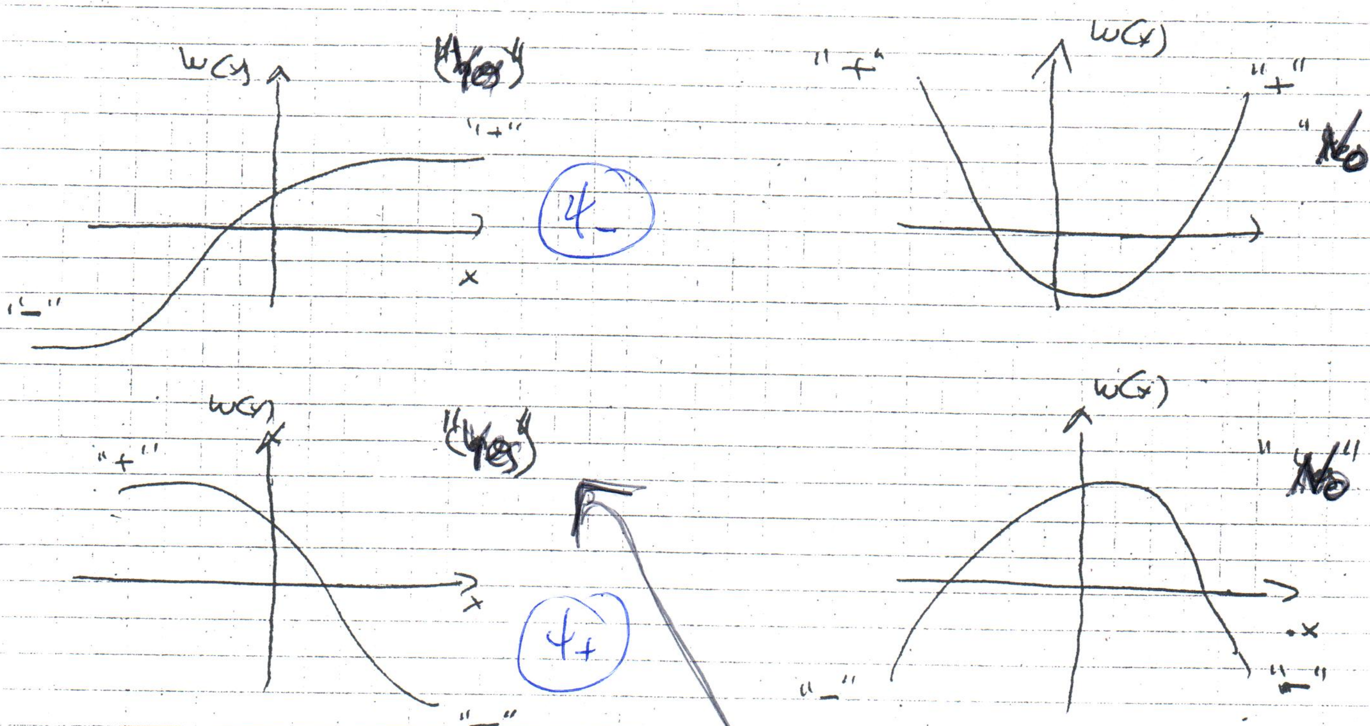
From this requirement we immediately get restriction for superpotential $W(x)$

"ψ_+" $\quad \int_0^x W(x')\,dx' \xrightarrow{x \to \pm\infty} -\infty$

"ψ_-" $\quad \int^x W(x')\,dx' \xrightarrow{x \to \pm\infty} +\infty$

(vacuum state is if any nondegenerate)

It is evident that only one of the two solutions can be normalized and the very existance of zero energy (vacuum) state depends only on the <u>global</u> properties of superpotential $W(x)$



~~For "kink"-shaped $W(x)$ there are no normalized vacuum solution (SUSY is spontaneously broken $E_0 > 0$)~~

$E_0 = 0$ (Exact SUSY)

7. Coherent States in Fock Space (Holomorphic representation)

Coherent states in Fock space play the role analogous to coordinate basis in Hilbert space. We have seen already that ~~there is a ...~~ two problems, namely, harmonic oscillators and bosonic states in Fock space are actually equivalent. In section 3 we introduced coherent states for quantum oscillator (coherent states are eigenstates of annihilation operator). In complete analogy with the harmonic oscillator problem one can define coherent states $|\varphi\rangle$ in Fock space. At first we consider boson case where the analogy is really complete

$$(56) \quad \underbrace{a_\alpha}_{\text{destruction operator in Fock space}} |\varphi\rangle = \underbrace{\varphi_\alpha}_{\text{complex number}} |\varphi\rangle \xleftrightarrow[\text{relation}]{\text{adjoint}} \langle\varphi| a_\alpha^+ = \langle\varphi|\varphi_\alpha^*$$

For single oscillator coherent state may be represented as follows (see p.15)

$$|\varphi\rangle \sim e^{\varphi a^+}|0\rangle \quad \Big\} \begin{array}{l}\text{we don't care about normalization} \\ \langle\varphi|\varphi\rangle \neq 1\end{array}$$

Generalization of this equation to a set of noninteracting oscillators (this problem is fully equivalent to the one for Fock space of bosons) is

$$(58) \quad |\varphi\rangle = e^{\sum_\alpha \varphi_\alpha a_\alpha^+}|0\rangle \quad \Big\} \text{ Notice that state is not normalized to unit}$$

$\Downarrow$

adjoint relation

$$\langle\varphi| = \langle 0| e^{\sum_\alpha \varphi_\alpha^* a_\alpha}$$

From Eq. (57) it is evident that

$$a_\alpha^+|\varphi\rangle = a_\alpha^+ e^{\sum_\beta \varphi_\beta a_\beta^+}|0\rangle = \frac{\partial}{\partial \varphi_\alpha}|\varphi\rangle \qquad (58)$$

$$\langle\varphi|a_\alpha = \langle\varphi|\frac{\overleftarrow{\partial}}{\partial\varphi_\alpha^*}$$

adjoint relation

We see that creation operator acts on ket-vector $|\varphi\rangle$ as momentum operator does on ket-vector $|\vec{r}\rangle$ (up to nonessential factor $-i$).

The overlap of two coherent states

$$\langle\varphi|\varphi'\rangle = e^{\sum_\alpha \varphi_\alpha^* \varphi'_\alpha}$$

these states are not orthogonal (as we know already)

$$e^A e^B = e^B e^A e^{[A,B]}$$ (prove as an exercise)

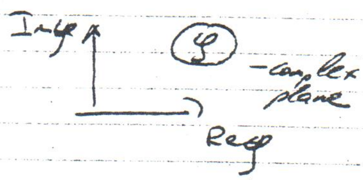
<u>Closure relation for coherent state in bosonic Fock space</u>

For single oscillator with normalized states $\langle\alpha|\alpha\rangle = 1$ the closure relation is (see Eq. 23)

$$\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = I \; , \quad \int \frac{d^2\varphi}{\pi} e^{-\varphi\varphi^*}|\varphi\rangle\langle\varphi| = I$$

unnormalized states

$$\frac{d^2\varphi}{\pi} = \frac{d\mathrm{Re}\varphi \, d\mathrm{Im}\varphi}{\pi} \Leftrightarrow \frac{d\varphi^* d\varphi}{2\pi i}$$



The last equality defines the measure of integration in terms of φ^*, φ. Factor $\frac{1}{2i}$ means, actually, that it is not ordinary product of differentials

$$\underbrace{dz^* \wedge dz}_{\text{wedge product of 1-(differential) forms}} = (dx - i dy)\wedge(dx + i dy) = -i dy\wedge dx + i dx \wedge dy = i2 dx\wedge dy$$

(like a vector product of ordinary vectors $\Rightarrow dx\wedge dx = dy \wedge dy = 0$

The desired closure relation for coherent states in Fock space looks like (Bosonic case)

$$(59)\quad \boxed{\int \ldots \int \prod_{\alpha} \underbrace{\frac{d\varphi_\alpha^* d\varphi_\alpha}{2\pi i}\, e^{-\sum_\alpha \varphi_\alpha^* \varphi_\alpha}}\; |\varphi\rangle\langle\varphi| = \hat{I}}$$

normalization factor can be included in the integration measure $D\varphi^* D\varphi$

Any state vector $|\psi\rangle$ in the Fock space can be expanded in coherent basis

$$|\psi\rangle = \int D\varphi^* D\varphi\, \underbrace{\langle\varphi|\psi\rangle}_{\substack{\text{III} \\ \psi(\varphi^*)}}\, |\varphi\rangle$$

$\Leftarrow$ Definition of wave function in coherent state basis

$\psi(\varphi^*)$ depends only on φ^*. It is holomorphic (analytic) function of complex variable φ. In holomorphic representation the matrix elements of annihilation and creation operators are

$$\langle\varphi|a_\alpha|\psi\rangle = \frac{\partial}{\partial\varphi_\alpha^*} \underbrace{\langle\varphi|\psi\rangle}_{\substack{\text{III} \\ \psi(\varphi^*)}} = \frac{\partial\psi(\varphi^*)}{\partial\varphi_\alpha^*}$$

$a^+|\varphi\rangle = \frac{\partial}{\partial\varphi}|\varphi\rangle$

$\langle\varphi| a = \langle\varphi|\frac{\partial}{\partial\varphi^*}$

$$\langle\varphi|a_\alpha^+|\psi\rangle = \varphi_\alpha^* \langle\varphi|\psi\rangle = \varphi_\alpha^* \psi(\varphi^*)$$

$\langle\varphi| a_\alpha^+ = \langle\varphi|\varphi_\alpha^*$

So the realization of Heisenberg-Weyl algebra in coherent state basis is

$$(60)\quad \left\{ \begin{array}{l} a_\alpha \Longleftrightarrow \dfrac{\partial}{\partial\varphi_\alpha^*} \\ a_\alpha^+ \Longleftrightarrow \varphi_\alpha^* \end{array} \right\}$$

Operators in holomorphic representations

① $\hat{n}_\alpha = a^+_\alpha a_\alpha \quad \Rightarrow \quad \varphi^*_\alpha \frac{\partial}{\partial \varphi^*_\alpha}$

② Hamiltonian. Schrödinger equation

$$\hat{H}(a^+_\alpha, a_\alpha)\,|\Psi\rangle = E|\Psi\rangle$$

$$\Downarrow$$

(61) $$H\left(\varphi^*_\alpha, \frac{\partial}{\partial \varphi^*_\alpha}\right)\Psi(\varphi^*) = E\,\Psi(\varphi^*)$$

If Hamiltonian is ~~the~~ sum of 1-body (kinetic energy) and 2-body (interaction) operations, ~~then~~ it is easy to get from Eq. (61)

$$\left\{ \sum_{\alpha,\beta}' T_{\alpha\beta}\, \varphi^*_\alpha \frac{\partial}{\partial \varphi^*_\beta} + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} (\alpha\beta|\hat{V}|\gamma\delta)\, \varphi^*_\alpha \varphi^*_\beta \frac{\partial}{\partial \varphi^*_\delta}\frac{\partial}{\partial \varphi^*_\gamma} \right\} \Psi(\varphi^*)$$
$$= E\,\Psi(\varphi^*)$$

Grassmann Algebra. Fermion ~~s~~ Coherent States

boson c.s.'s were introduced by Schrödinger as a nondispersive wave packets

Now our goal is to construct fermion coherent states. We know that for boson fields coherent states minimize Heisenberg ~~s~~ uncertainty relation and are analogous to classical distribution function in phase space.
Fermion fields have no classical analog. It is impossible to measure fermion variable — only bilinear combinations of fermion fields can be measured. So if we are going to ~~[illegible]~~ introduce somehow eigenstate of fermion operator we could expect difficulties from

from the very beginning

$$\hat{c}|\alpha\rangle = \varsigma_\alpha|\alpha\rangle \Rightarrow \hat{c}^2|\alpha\rangle = \varsigma_\alpha^2|\alpha\rangle = 0$$

(Pauli's principle

$$\Downarrow$$

$$\varsigma_\alpha^2 = 0 \Rightarrow \varsigma_\alpha = 0 \; !$$

(if ς is ordinary number)

Such a trivial solution (the only one possible for c-numbers) does not satisfy us. Hence to introduce fermion coherent states one should make use of anticommuting "numbers"

(62) $$\xi_\alpha \xi_\beta + \xi_\beta \xi_\alpha = 0 \xRightarrow{\alpha=\beta} \xi_\alpha^2 = 0$$

In mathematics such an algebra is known as Grassmann algebra (ξ_α are called generators of GA and the dimension, i.e. the number of independent elements of GA with "n" generators, ~~the~~ is 2^n)

Introduce also operation (which we will denote by "$*$") analogous to complex conjugation "asteric"

$$(\xi_\alpha)^* = \xi_\alpha^* \quad \text{— is the new independent set of generators}$$

$n = 2$

Generators: ξ, ξ^*

(63) dimension: $2^2 = 4$: $\underbrace{1, \xi, \xi^*, \xi^*\xi}_{\text{elements of } n=2 \text{ GA}}$ $\quad \boxed{\xi\xi^* = -\xi^*\xi}$

Any "analytic" function ~~of~~ of Grassman variable ξ (or ξ^*)

is a linear function

(64) $f(\xi) = f_0 + f_1\xi \qquad\qquad f(\xi^*) = f_0^* + f_1^*\xi^*$

ordinary numbers $\qquad\qquad$ ordinary numbers

A function $A(\xi^*,\xi)$ has four terms in expansion in series of ξ, ξ^*

$$A(\xi^*,\xi) = a_0 + a_1\xi + a_2\xi^* + a_{12}\xi^*\xi$$

Now we are ready to introduce fermion coherent states as the eigenstates of fermion annihilation operator

(65) $\boxed{\hat{c}_\alpha|\xi\rangle = \xi_\alpha|\xi\rangle}$ $\qquad \xi_\alpha$ is the Grassmann "number"

We postulate aslo anticommutation relations between Grassman "numbers" and fermion operators $\hat{c}_\alpha, \hat{c}_\alpha^+$

(66) $\boxed{\{\xi_\alpha, \hat{c}_\beta\} = 0 \qquad (\xi_\alpha \hat{c}_\beta)^* = \hat{c}_\beta^+ \xi_\alpha^*}$

Then in the full analogy with the boson case fermion coherent states can be created from vacuum $|0\rangle$ by repeating action of creation operators

notice "−" sign

(67) $$|\xi\rangle = e^{-\sum_\alpha \xi_\alpha \hat{c}_\alpha^+}|0\rangle = \prod_\alpha e^{-\xi_\alpha \hat{c}_\alpha^+}|0\rangle = \prod_\alpha (1-\xi_\alpha \hat{c}_\alpha^+)|0\rangle$$

$[\xi_\alpha \hat{c}_\alpha^+, \xi_\beta \hat{c}_\beta^+] = 0$

$\Downarrow$ adjoint rel.

$$\langle\xi| = \langle 0|e^{\sum_\alpha \xi_\alpha^* \hat{c}_\alpha} \qquad\qquad \langle\xi|\hat{c}_\alpha^+ = \langle\xi|\xi_\alpha^*$$

It is easy to check that $|\xi\rangle$ really is an eigenstate of annihilation operator $\hat{c}_\alpha$. Let for simplicity $\xi_\alpha = \xi$

$$\hat{c}_\alpha|\xi\rangle = \hat{c}(1-\xi_\alpha\hat{c}^+)|0\rangle = \underbrace{c|0\rangle}_{0} + \underbrace{\xi c c^+|0\rangle}_{\{\xi, c\}=0} = \xi(1-c^+c)|0\rangle = \xi|\xi\rangle$$

$\{c, c^+\}=1$

$\xi|\xi\rangle = \xi(1-\xi\hat{c}^+)|0\rangle = \xi|0\rangle$

To define how $\hat{c}^+$ acts on $|\xi\rangle$ we should introduce the derivative upon Grassmann "number"

Definition.

$$\frac{\partial A(\xi^*,\xi)}{\partial\xi} = \frac{\partial}{\partial\xi}\left(a_0 + \xi a_1 + a_2\xi^* + \underbrace{a_{12}\xi^*\xi}_{-a_{12}\xi\xi^*}\right) = a_1 - a_{12}\xi^*$$

$$\frac{\partial A(\xi^*,\xi)}{\partial\xi^*} = a_2 + a_{12}\xi$$

Hence $\frac{\partial}{\partial\xi^*}\frac{\partial}{\partial\xi} = -a_{12} = -\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi^*} \Rightarrow \boxed{\left\{\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi^*}\right\}=0}$

By making use of above rules one immediately gets

(68)
$$\begin{cases} \hat{c}^+_\alpha|\xi\rangle = \hat{c}^+_\alpha e^{-\sum'_\alpha \xi_\alpha \hat{c}^+_\alpha}|0\rangle = -\frac{\partial}{\partial\xi_\alpha}|\xi\rangle \\ \langle\xi|\hat{c}_\alpha = \langle 0|e^{-\sum'_\alpha \hat{c}_\alpha\xi^*_\alpha}\hat{c}_\alpha = \langle 0|e^{+\sum'_\alpha \xi^*_\alpha\hat{c}_\alpha}\hat{c}_\alpha = \frac{\partial}{\partial\xi^*_\alpha}\langle\xi| \end{cases}$$

The overlap of two coherent states is

$$\langle\xi|\xi'\rangle = \langle 0|\prod_{\alpha\beta}(1+\xi^*_\alpha\hat{c}_\alpha)(1-\xi'_\beta\hat{c}^+_\beta)|0\rangle =$$

$$= \langle 0|\prod_{\alpha,\beta}(1 + \cancel{\xi^*_\alpha\hat{c}_\alpha} - \cancel{\xi'_\beta\hat{c}^+_\beta} + \xi^*_\alpha\xi'_\beta(\delta_{\alpha\beta} - \cancel{\hat{c}^+_\beta\hat{c}_\alpha}))|0\rangle =$$

$$= \underbrace{\langle 0|0\rangle}_{1}\prod_\alpha(1+\xi^*_\alpha\xi'_\alpha) = e^{+\sum'_\alpha\xi^*_\alpha\xi'_\alpha}$$

(Compare with boson case $\langle\varphi|\varphi'\rangle = e^{+\sum'_\alpha\varphi^*_\alpha\varphi'_\alpha}$

To complete the description of fermion coherent state we need to derive closure relation for $|\xi\rangle$

For boson case the measure is $D\varphi^* D\varphi \equiv \frac{d^2\varphi}{\pi} e^{-|\varphi|^2}$

Now the questions arise! — (i) what is the integration over Grassmann "numbers"?

(ii) what is the measure of integration for coherent states?

(i) We have no experience in integration of such ~~strange~~ exotic objects as Grassmann numbers. What is infinitely small number $d\xi$? It does not exist in ordinary sense. The only way out is to postulate integration rules. They should be internally unambiguous. These rules were proposed by Berezin.

$$\left.\begin{aligned} &\int d\xi = 0 \qquad &\int d\xi^* = 0 \\ &\int d\xi\, \xi = 1 \qquad &\int d\xi^* \xi^* = 1 \end{aligned}\right\} \tag{69}$$

Berezin's rules of integration

(Mnemonic rule: integration $=$ differentiation)

Double integrals are regarded as repeated ones.

Examples:

1) $\int d\xi f(\xi) = \int d\xi (f_0 + f_1 \xi) = f_1$

$\int d\xi A(\xi^*, \xi) = \int d\xi \left(a_0 + a_1 \xi + a_2 \xi^* + a_{12} \xi^* \xi\right) = a_1 - a_{12}\xi^*$

$\int d\xi^* A(\xi^*, \xi) \qquad = a_1 + a_{12}\xi$

$\int d\xi^* d\xi A(\xi^*, \xi) = -a_{12} = -\int d\xi d\xi^* A(\xi^*, \xi) \Rightarrow \{d\xi, d\xi^*\} = 0$

2) Grassmann δ-function

$$\delta(\xi,\xi') = \int d\eta\, e^{+\eta(\xi-\xi')} = \int d\eta\left[1+\eta(\xi-\xi')\right] = +(\xi-\xi')$$

$$\int d\xi'\,\delta(\xi,\xi')f(\xi') = +f(\xi)$$

Berezin's rules for Grassmann integrals looks from the first glance very unnatural. How is it possible to guess that this is only correct rules of integration over Grassmann number?

I try to motivate the above rules.

Let us consider the scalar product of two Grassmann functions f and g

$$\langle f|g\rangle = \int d\xi^* d\xi\, W_F(\xi^*,\xi)\, f^*(\xi)\, g(\xi^*)$$

the weight function will provide proper normalization.

We know that in boson case $W_b(\varphi,\varphi^*) = e^{-\varphi^*\varphi}$.

Let us keep this function for fermion case as well $W_F = e^{-\xi^*\xi}$.

In boson case we have ~~other~~ evident property

(*) $\langle 0|0\rangle_B = 1 = \langle 1|1\rangle \qquad \langle 0|1\rangle_B = \langle 1|0\rangle_B = 0$

If we assume that ~~the orthogonal~~ orthonormality conditions (*) hold also for fermions, then Berezin's integration rules are derived automatically.

Closure relation for fermion coherent states (Problem)

$$(70)\quad \boxed{\int \prod_\alpha d\xi_\alpha^* d\xi_\alpha\, e^{-\sum_\alpha \xi_\alpha^*\xi_\alpha}\, |\xi\rangle\langle\xi| = \hat{I}}$$

Prove it by direct calculation

The (over-)completeness of fermion coherent states allows us to determine state vector in holomorphic representation

$$|\Psi\rangle = \int \prod_\alpha d\xi^*_\alpha d\xi_\alpha \, e^{-\sum_\alpha \xi^*_\alpha \xi_\alpha} \Psi(\xi^*) |\xi\rangle$$

$$\Psi(\xi^*) \equiv \langle \xi | \Psi \rangle$$

Then

(21) $$\langle \xi | c_\alpha | \Psi \rangle = \frac{\partial}{\partial \xi^*_\alpha} \Psi(\xi^*)$$

$$\langle \xi | c^+_\alpha | \Psi \rangle = \xi^*_\alpha \Psi(\xi^*)$$

$$\Longrightarrow \underbrace{\left\{ \frac{\partial}{\partial \xi^*_\alpha}, \xi^*_\beta \right\} = \delta_{\alpha\beta}}_{}$$

"Holomorphic representations" of algebra for fermion operators.

Remark

Boson coherent states belong to Fock space and can be measured. For instance for photons

$$|\varphi\rangle = e^{\int d\vec{r}\, \varphi(\vec{r}) \hat{\Phi}^+(\vec{r})} |0\rangle$$

$\varphi(\vec{r})$ is nothing but the amplitude of electromagnetic field.

Fermion coherent state do not belong to Fock space. (They belongs to generalized Fock space with Grassmann "coordinates"). One can not "measure" Grassmann function. However Grassmann fields are very useful in the path integral formalism.